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Measures of Maxima & Minima of a Cauchy Random Polynomial

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ABSTRACT

The aim of this paper is to estimate the number of real zeros of a Cauchy random polynomial under different condition when the coefficients belong to the domain of attraction of Cauchy law. Let $\sum_{r=0}^{n} G_r(X)W^r$ be a Cauchy random algebraic polynomial of degree n, whose coefficients G_r (X)'s identically distributed independent random variables belonging to the domain of attraction of Cauchy distribution. Then there exists a positive integer n_0, such that for $n>n_0$, the number of real roots of most of the equation $\sum_{r=0}^{n} G_r(X)W^r$ =0 is atmost μ (log n) ^{2} except for a set of measure atmost $\frac{\mu'}{n_0^{1-s}}$, where **μ** and **μ'** are positive constants and $0<s<1$.

Introduction:

Let N_n (X) be the number of maxima & minima of the random algebraic polynomial $\sum_{r=0}^{n} G_r(X)W^r$, where the coefficients y_r (u)' s are independent random variables. Samal [3] has considered the general case when G_r (X)'s are identically distributed normal variates with mean zero, variance one and third absolute moment finite and non-zero. Evans [1] has studied the strong version of the upper bound of N_n (X) when the coefficients are normal variates. Samal and Mishra [4] have considered the upper bound N_n (X) when the coefficients are identically distributed symmetric stable varieties. But we give an estimate for an upper bound of N_n (X) both in weak and strong sense when the coefficients belong to the domain of attraction of Cauchy distribution. Mishra, Nayak and Pattnayak[2] consider the lower bounds of the number of real roots of a random algebraic polynomial and get a better result . In the following theorems we give an estimate for upper bound of the number of real zeros of a random polynomial, when the coefficients belong to the domain of attraction of Cauchy law both in strong and weak1 sense.

1.2. Theorem 1:

Let $\sum_{r=0}^{n} G_r(X)W^r$ be a random algebraic polynomial of degree n, whose coefficients G_r (X)'s are identically distributed independent random variables belonging to the domain of attraction of Cauchy distribution. Then there exists a positive integer n_0, such that for $n>n_0$, the number of real roots of most of the equation $\sum_{r=0}^{n} G_r(X)W^r$ =0 is atmost $\mu(\log n)^2$ except for a set of measure atmost $\frac{\mu'}{n_0^{1-s}}$, where μ and μ' are positive constants and $0<s<1$.

1.3. Theorem 2:

Let $\sum_{r=0}^{n} G_r(X)W^r$ be a random algebraic polynomial of degree n, whose coefficients G_r (X)'s are identically distributed independent random variables belonging to the domain of attraction of Cauchy distribution. Then there exists a positive integer n_0, such that for $n>n_0$, the number of real roots of most of the equation $\sum_{r=0}^{n} G_r(X)W^r=0$ is atmost $\mu(\log n)^2$ except for a set of measure atmost $\dfrac{\mu'}{n_0^{1-s}}$, here μ and μ' are positive constants and $0<s<1$.

1.4. Corresponding to each root of f(x) =0 in (0, 1). There exists a root of f (-x) =0 in (-1, 0) and conversely. Again f(x) =0 has a root in (1, ∞) implies $x^n f(G)=0$ has root in (0, 1) where G=1/x. Sambadham [5] consider the number of real zeros of a random algebraic polynomial in the interval (0, 1). The number of zeros in the interval (-∞, ∞) and the measure of the exceptional set will be each four times of the corresponding estimates for the range (0, 1). We choose a fixed number p greater than 1/log 2 and let k= [p log n], where [p log n] denotes the greatest integer not exceeding p log n. We consider circles Cm, m= 1, 2…….k p log n with center $x_m=\left(1-\frac{1}{2^m}\right)$ and radius

$r_m=\frac{1}{2}(1-x_m)=\frac{1}{2}\left(\frac{1}{2^m}\right)=\left(\frac{1}{2^{m+1}}\right)$ and in the special case $r_0=1/n$ when $x_0=1$. The circles $C_0, C_1,\ldots\ldots\ldots\ldots C_k, C_p$ log n cover the closed segment [1/2, 1]. Let Γ_m be the circles concentric with C_m and of double the radius C_m. Therefore all Γ_m are interior to $z=\left(1+\frac{2}{n}\right)$. The characteristics function Φ(t) of a distribution belonging to the domain of attraction of the Cauchy distribution is given by

$$\Phi(t)=\exp(-C|t|h(t))$$

where h(t) is a slowly varying function as t→∞

or $\quad \Phi(t)=\exp(-C|t|H(t))$

where H(t) is a slowly varying function as t→0.

1.5. We need the following lemmas for the proof of the theorem.

Lemma 1.1

If G(X) is identically distributed independent random variables belonging to the domain of attraction of the Cauchy distribution, then

$$P\{|G(X)|\le\eta\}<\frac{2\eta}{\pi}\left\{\frac{\Gamma\left(\frac{1}{1+1}\right)}{(1+s)C^{\frac{1}{1+s}}}+\mu\,\frac{\Gamma\left(\frac{1}{1-s}\right)}{{}_1(1-s)C^{\frac{1}{1-s}}}\right\}$$

Proof: This is the direct consequence of the paper Mishra, Nayak and Pattnayak [2] since the exponent 'α' of stable law takes value 1 in Cauchy law.

Lemma 1.2 If the random variable y(u) belong to the domain of attraction of Cauchy law, then for s>0,

$$P\{|G(X)|\le\eta\}\le<\frac{\mu}{\eta^{1-s}}$$

Proof: This is also a direct consequence of the paper Mishra, Nayak and Pattnayak [2] by putting α=1.

1.6. Proof of the theorem.

Let us consider the circle C_m with $Z_0=x_m$, $R=2r_m$, $r=r_m$ and $N_m(u)$ be the number of real zeros of

$$\sum_{r=0}^{n} G_r(X)W^r \qquad (1.1)$$

Then applying lemma 1.2 we have,

$$N_m(X)<\frac{1}{\log 2}\log\left\{\frac{z\max_{\le 1+\frac{2}{n}}|G(z)|}{G(x_m)}\right\} \qquad (1.2)$$

By using lemma 1.1 we have for s>0,

$$P\{|G_r(X)|>(n+1)^3,\ 0\le r\le n\}$$

$$=\frac{\mu}{(n+1)^{2-s}} \qquad (\text{for } s=3s)$$

Thus $P\{|y_r(u)|\le(n+1)^3,\ 0\le r\le n\}$(1.3)

$$>1-\frac{\mu}{(n+1)^{2-s}} \qquad (1.4)$$

If G(z) is a regular function in a circle with center z_0 and of radius 'r', if follows from lemma 1.2 that outside a set a measure atmost

$$\frac{\mu}{(n+1)^{2-s}} \qquad (1.5)$$

Now using lemma 1.1, for s>0, we have

Therefore $\quad |G_r(X)|\le\frac{1}{n^5} \qquad (1.6)$

except for a set of measure atmost

$$\frac{2}{\pi n^5}\left\{\frac{\Gamma\left(\frac{1}{\beta}\right)}{(\beta)C^{\frac{1}{\beta}}}+\mu\,\frac{\Gamma\left(\frac{1}{\nu}\right)}{{}_1\,\nu C^{\frac{1}{\nu}}}\right\}(1.7)$$

(Putting β=1+s, ν=1-s)

Again it follows from above

$$P\left\{|G(x_0)|\le\frac{1}{n^5}\right\}$$

$$\le\frac{\mu}{n^5}\left\{\left(\sum_{r=0}^{n}x^{r\beta}{}_0\right)^{-1/\beta}+\left(\sum_{r=0}^{n}x^{r\nu}{}_0\right)^{-1/\nu}\right\}(1.8)$$

$$\le\frac{\mu}{n^5}\left\{\frac{1}{(n+1)^{1/\beta}}+\frac{1}{(n+1)^{1/\nu}}\right\}$$

$$<\frac{\mu'}{n^{5+1/\beta}}(1.9)$$

Therefore, $\quad |G(x_0)|\le\frac{1}{n^5} \qquad (1.10)$

except a set of measure atmost

$$\frac{\mu'}{n^{5+1/\beta}} \qquad (1.11)$$

Hence using (1.4), (1.5), (1.9) we get from (1.2) that outside a set of measure atmost

$$\frac{\mu}{(n+1)^{2-s}}+\frac{\mu}{n^5}\left\{\left(\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta}\right)^{-1/\beta}+\left(\left(1-\frac{1}{2^m}\right)^{r\nu}\right)^{-1/\nu}\right\}$$

The number of zeros of f (x) in C_m is atmost

$$\frac{\log(e^2(n+1)^4n^5}{\log 2}=\frac{2+4\log(n+1)+5\log n\mu}{\log 2}$$

$$=\mu\log n,\text{for}\, m>0.$$

Again using (1.4), (1.5), (1.11) we get from (1.2) that outside a set of measure atmost

$$\frac{\mu}{(n+1)^{2-s}}+\frac{\mu'}{n^{5+\frac{1}{\beta}}}$$

the number of zeros of f(x) in C_0, is atmost

$$\frac{\log(e^2(n+1)^4n^5}{\log 2}=\mu\log n$$

Considering the circles C_0, C_1, C_2...........C_k, C_p log n the total number of zeros inside all the circles is atmost

$$\mu(k+1)\log n<\mu_1(\log n)^2 \quad (1.12)$$

except for a set of measure atmost

$$\frac{\mu'}{n^{5+1/\beta}} \quad (1.13)$$

1.6. Let ,

$$L_1=\sum_{m=1}^{\frac{\log n}{\log 2}}\left\{\left(\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta}\right)^{-1/\beta}+\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta-1/\nu}\right\}$$ Now we have,

$$=\sum_{2\,\frac{\log n}{\log 2}+1}^{\frac{\log n}{\log 2}}\left(\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta}\right)^{-1/\beta}+\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta-1/\nu}$$

Again using the inequality $(1-x)^n<1-nx$ when

$$n<0 \text{ or } n>1 \text{ for } |x|<1, r\neq 0$$

we have

$$\left(1-\frac{1}{2^m}\right)^{\beta}\geq\left(1-\frac{\beta}{2^m}\right)$$

$$\frac{\beta}{2^m} \quad (1.16)$$

Therefore from (1.15), (1.16) we have, in S_1,

$$\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta}\geq\frac{1-\left(1-\frac{1}{n}\right)^{\beta n}}{\left(\frac{\beta}{2^m}\right)}$$

Now taking the 1st part of S_1,

$$\sum_{m=1}^{\frac{\log n}{\log n}}\left\{\left(\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta}\right)^{-1/\beta}\right\}$$

$$<\sum_{m=1}^{\frac{\log n}{\log n}}\left\{\frac{1-\left(1-\frac{1}{n}\right)^{n}}{\beta/2^m}\right\}^{-1/\beta}$$

(Proceeding similarly as 1st part of S_1)

So

$$\frac{S_1}{n^5}<\frac{1}{n^5}(\mu_1+\mu_2)=\frac{\mu}{n^5} \quad (1.19)$$

Now taking the 1st part of S_2

$$\sum_{\frac{\log n}{\log 2}+1}^{p\log n}\left\{\left(\sum_{r=0}^{n}\left(1-\frac{1}{2^m}\right)^{r\beta}\right)^{-1/\beta}\right\}$$

$$<\sum_{\frac{\log n}{\log 2}+1}^{p\log n}\left\{n\left(n-\frac{1}{n}\right)^{\beta n}\right\}^{-1/\beta}$$

$$\leq\frac{\log n}{n^{1/\beta}e^{-1/D}} \qquad (\text{where } D>1)$$

$$<\frac{\mu'\log n}{n^{1/\beta}} \qquad (1.20)$$

Now taking the 2nd part of S_2

So by the help of (1.19) and (1.22) we get from (1.13) that, the measure of exceptional set is atmost.

1.7. Now applying the procedure of Samal and Mishra [4] we consider the sequent (0,1/2) we take a circle with center zero and radius ½.

The circle $|z|\leq\frac{1}{2}$ is interior to the circle $|z|\leq 1$.

Applying lemma 1.1 with $z_0=0$, r=1/2, R=1, we have the number of zeros of f(x) in the circle C i.e. $|z|\leq\frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left\{\max_{|z|\leq 1}\frac{|f(z)|}{|f(0)|}\right\} \qquad (1.24)$$

we know from (4.3) that

$$P\{|y_r(u)|\leq(n+1)^3, 0\leq r\leq n\}>1-\frac{\mu}{(n+1)^{2-s}}$$

So outside a set of measure atmost

$$\frac{\mu}{(n+1)^{2-s}}(1.25)$$

We have,

$$\max_{|z|\leq 1}|f(z)|\leq\max_{|z|\leq 1}\sum_{r=0}^{n}|y_r(u)||z|^r$$

$$\leq\sum_{r=0}^{n}(n+1)^3 1^r$$

$$\leq\mu(n+1)^4 \qquad (1.26)$$

By using lemma 1.1 we obtain that

$$P\left\{|f(o)|\leq\frac{1}{n^5}\right\}<\frac{\mu\log n}{n^{2-s}} \qquad (1.27)$$

[Since 2-s<5, for s>0 and log n<n $1/n^5<\log n/(n)^{2-s}$]

So using (1.25), (1.26), (1.27) we get from (1.24) that the number of zeros inside the circle C' does not exceed

$$\frac{\log\mu n^9}{\log 2}<\mu\log n<\mu(\log n)^2 n \quad (1.28).$$

Outside a set of measure atmost

$$\frac{\mu}{(n+1)^{2-s}}+\frac{\mu\log n}{n^{2-s}}<\frac{\mu\log n}{n^{2-s}} \quad (1.29)$$

1.8 In case $y_0(u)$ and hence f(o) is zero with positive probability, we take a circle with center $\sigma\left(0<\sigma<\frac{1}{2}\right)$ and radius ½ .Thus the circle $|z-\sigma|\leq(1-\sigma)$ is interior to the circle $|z|\leq(1)$. Now considering the procedure of Samal and Mishra [4] and applying lemma 1.1 with $z_r=\sigma, r=\frac{1}{2}$, R=(1-0), the number of zeros of f(x) in the circle C'', i.e. $|z-\sigma|\leq(1/2)$ does not exceed

$$\frac{\log \max\limits_{|z|\leq 1} \frac{|f(z)|}{f(0)}}{\log\{2(1-\sigma)\}} \quad (1.30)$$

We know from (1.25) and (1.26) that outside a set of measure atmost

$$\frac{\mu}{(n+1)^{2-s}}<\frac{\mu}{(n)^{2-s}} \quad (1.31)$$

$$\max_{|z|\leq 1}|f(z)|\leq\mu(n+1)^4 \quad (1.32)$$

By using lemma 1.2, we have

Therefore

$$|f(\sigma)|\geq\frac{1}{n^5},$$

except for a set of measure atmost

$$\frac{\mu\log n}{n^{2-s}} \quad (1.34)$$

So by using (1.31), (1.32), (1.34) we obtain from (1.30) that the number of zeros inside the circle C'' does not exceed

$$\frac{\log\mu n^9}{\log 2(1-\sigma)}<\mu\log n<\mu(\log n)^2 \quad (1.35)$$

Outside a set of measure atmost

$$\frac{\mu}{(n+1)^{2-s}}+\frac{\mu\log n}{n^{2-s}}<\frac{\mu\log n}{n^{2-s}} \quad (1.36)$$

1.9 Now let us consider the number of real zeros N_n in the whole interval (0,1). From the sections (1.5), (1.7), (1.8) that $N_n<(\log n)^2$.

Outside a set of measure atmost

$$\sum_{n0}^{\infty}\frac{\mu'\log n}{n^{2-s}}<\frac{\mu'}{n^{2-3s}}$$

(since for large n, log n<n^{2s})

$<\frac{\mu'}{n_0^{1-s}}$, where 1>s>0.

1. 10. Proof of the theorem 1.2:

We cover the closed segment [1/2, 1] by the circles mentioned in the section (1.3), we have already established in (1.3) that

$$P\{|y_r(u)|\leq(n+1)^3, 0\leq r\leq n\}$$

$$>1-\frac{\mu}{(n+1)^{2\text{-}s}} \quad (1.37)$$

Therefore except a set of measure atmost

$$\frac{\mu}{(n+1)^{2\text{-}s}} \quad (1.38)$$

We have

$$\max_{|z|\leq 1+2/n}|f(z)|\leq\sigma^2(n+1)^4 \quad (1.39)$$

By using lemma (1.1), we get that

(1.40)

where β=1+s, ν=1-s

or, $\left\{|y_r(u)|\geq\frac{1}{n}\right\}$ (1.41)

except for a set of measure atmost

So

except for a set of measure atmost $\frac{1}{n}\mu$

for m=1,2,3,……….k, p log n.

Similarly

[Since 1/β<1/ν] or $|f(x_m)|\geq 1/n$

except for a set of measure atmost

$$\frac{\mu}{n^{1+1/\beta}} \quad (1.45)$$

By using (1.38), (1.39), (1.43) we get from (1.2) that outside a set of measure atmost

The number of zeros of f(x) in C_m is atmost

$$\frac{\log(e^2(n+1)^3 n}{\log 2}<\mu\log n, \text{for } n>0$$

Using (1.38), (1.39), (1.44) we get from (1.2) that outside a set of measure atmost

$$\left\{\frac{\mu}{(n+1)^{2-s}}+\frac{\mu}{n^{1+1/\beta}}\right\}$$

The number of zeros of f(x) in C_0 is atmost μ log n, for m=0. Considering all the circles C_0, C_1, C_2…….C_k, C_p log n the total number of zeros inside all the circles is atmost.

$$\mu(k+2)\log n<\mu(\log n)^2 \quad (1.46)$$

except for a set of measure atmost

$$=\frac{\mu}{n^{1+\frac{1}{\beta}}} \quad (1.47)$$

1.11. Consider the sum

$$=\frac{1}{n}(S_1+S_2)$$

Following the procedure adopted in section 1.6, we get

Therefore from (1.47) we obtain that the measure of the exceptional set is atmost (Since log n<n^s for large n, 0<s<1)

1.12

It has been calculated in theorem 1.1, section 1.1.7 and 1.1.8 that the number of zeros in the segment (0,1/2) does not exceed $\mu(\log n)^2$. Outside a set of measure atmost

$$\frac{\mu\log n}{n^{2-s}}$$

when is less than $\frac{\mu}{n^{1-s}}$.

Therefore considering the number of real zeros in the whole interval (0, 1) $N_n<\mu(\log n)^2$.

Outside a set of measure atmost $\frac{\mu}{n^{1-s}}$ where s>0.

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